HAMILTONIAN MINIMAL LAGRANGIAN SUBMANIFOLDS IN TORIC VARIETIES

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Hamiltonian minimality (H-minimality for short) for Lagrangian submanifolds is a symplectic analogue of Riemannian minimality. A Lagrangian immersion is called H-minimal if the variations of its volume along all Hamiltonian vector fields are zero. This notion was introduced in the work of Y.-G. Oh [4] in connection with the celebrated *Arnold conjecture* on the number of fixed points of a Hamiltonian symplectomorphism.

In [2] and [3] the authors defined and studied a family of H-minimal Lagrangian submanifolds in $\mathbb{C}^m$ arising from intersections of real quadrics. Here we extend this construction to define H-minimal submanifolds in toric varieties.

The initial data of the construction is an intersection of $m-n$ Hermitian quadrics in $\mathbb{C}^m$:

$$\mathcal{Z}=\Bigl\{\boldsymbol{z}=(z_1,\ldots,z_m)\in\mathbb{C}^m\colon \sum_{k=1}^{m}\gamma_{jk}|z_k|^2=\delta_j\quad\text{for } j=1,\ldots,m-n\Bigr\}. \tag{1}$$

We assume that the intersection is nonempty, nondegenerate and rational; these conditions can be expressed in terms of the coefficient vectors $\gamma_i=(\gamma_{1i},\ldots,\gamma_{m-n,i})^t\in\mathbb{R}^{m-n}$, $i=1,\ldots,m$, as follows:

(a) $\delta\in\mathbb{R}_{\geqslant}\langle\gamma_1,\ldots,\gamma_m\rangle$ (δ is in the cone generated by $\gamma_1,\ldots,\gamma_m$);
(b) if $\delta\in\mathbb{R}_{\geqslant}\langle\gamma_{i_1},\ldots\gamma_{i_k}\rangle$, then $k\geqslant m-n$;
(c) the vectors $\gamma_1,\ldots,\gamma_m$ span a lattice L of full rank in $\mathbb{R}^{m-n}$.

Under these conditions, $\mathcal{Z}$ is a smooth $(m+n)$-dimensional submanifold in $\mathbb{C}^m$, and

$$T_\Gamma=\bigl\{\bigl(e^{2\pi i\langle\gamma_1,\varphi\rangle},\ldots,e^{2\pi i\langle\gamma_m,\varphi\rangle}\bigr),\quad\varphi\in\mathbb{R}^{m-n}\bigr\}=\mathbb{R}^{m-n}/L^*$$

is an $(m-n)$-dimensional torus. We represent elements of T_Γ by $\varphi\in\mathbb{R}^{m-n}$. We also define

$$D_\Gamma=(\tfrac12L^*)/L^*\cong(\mathbb{Z}_2)^{m-n}.$$

Note that D_Γ embeds canonically as a subgroup in T_Γ.

Let $\mathcal{R}\subset\mathcal{Z}$ be the subset of real points, which can be written by the same equations in real coordinates:

$$\mathcal{R}=\Bigl\{\boldsymbol{u}=(u_1,\ldots,u_m)\in\mathbb{R}^m\colon \sum_{k=1}^{m}\gamma_{jk}u_k^2=\delta_j\quad\text{for } j=1,\ldots,m-n\Bigr\}.$$

We 'spread' $\mathcal{R}$ by the action of T_Γ, that is, consider the set of T_Γ-orbits through $\mathcal{R}$. More precisely, we consider the map

$$\begin{aligned}j\colon\mathcal{R}\times T_\Gamma&\longrightarrow\mathbb{C}^m,\\(\boldsymbol{u},\varphi)&\mapsto\boldsymbol{u}\cdot\varphi=\bigl(u_1e^{2\pi i\langle\gamma_1,\varphi\rangle},\ldots,u_me^{2\pi i\langle\gamma_m,\varphi\rangle}\bigr)\end{aligned}$$

and observe that $j(\mathcal{R}\times T_\Gamma)\subset\mathcal{Z}$. We let D_Γ act on $\mathcal{R}_\Gamma\times T_\Gamma$ diagonally; this action is free since it is free on the second factor. The quotient

$$N=\mathcal{R}\times_{D_\Gamma}T_\Gamma$$

is an m-dimensional manifold.

Theorem 1 ([2]). *The map $j\colon\mathcal{R}\times T_\Gamma\to\mathbb{C}^m$ induces an H-minimal Lagrangian immersion $i\colon N\looparrowright\mathbb{C}^m$.*

Intersection of quadrics (1) is invariant with respect to the diagonal action of the standard torus $\mathbb{T}^m\subset\mathbb{C}^m$. The quotient $\mathcal{Z}/\mathbb{T}^m$ is identified with the set of nonnegative solutions of the system of linear equations $\sum_{k=1}^m\gamma_ky_k=\delta$. This set may be described as a convex n-dimensional polyhedron

$$P=\bigl\{\boldsymbol{x}\in\mathbb{R}^n\colon\langle\boldsymbol{a}_i,\boldsymbol{x}\rangle+b_i\geqslant0\quad\text{for } i=1,\ldots,m\bigr\}, \tag{2}$$

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where $(b_1, \dots, b_m)$ is any solution and the vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m \in \mathbb{R}^n$ form the transpose of a basis of solutions of the homogeneous system $\sum_{k=1}^m \gamma_k y_k = \mathbf{0}$. We refer to P as the *associated polyhedron* of the intersection of quadrics (1). The vector configurations $\gamma_1, \dots, \gamma_m$ and $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m$ are *Gale dual.*

Let N denote the lattice of rank n spanned by $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m$. Polyhedron (2) is called *Delzant* if, for any vertex $\boldsymbol{x} \in P$, the vectors $\boldsymbol{a}_{i_1}, \dots, \boldsymbol{a}_{i_k}$ normal to the facets meeting at $\boldsymbol{x}$ form a basis of the lattice N. A Delzant n-polyhedron is *simple*, that is, there are exactly n facets meeting at each of its vertices.

Theorem 2 ([3])**.** *The immersion $i\colon N \looparrowright \mathbb{C}^m$ is an embedding of an H-minimal Lagrangian submanifold if and only if the associated polyhedron P is Delzant.*

Now we consider two sets of quadrics:

$$\mathcal{Z}_\Gamma = \Big\{\boldsymbol{z} \in \mathbb{C}^m\colon \sum\nolimits_{k=1}^m \gamma_k |z_k|^2 = \boldsymbol{c}\Big\}, \quad \gamma_k, \boldsymbol{c} \in \mathbb{R}^{m-n};$$
$$\mathcal{Z}_\Delta = \Big\{\boldsymbol{z} \in \mathbb{C}^m\colon \sum\nolimits_{k=1}^m \delta_k |z_k|^2 = \boldsymbol{d}\Big\}, \quad \delta_k, \boldsymbol{d} \in \mathbb{R}^{m-\ell};$$

such that $\mathcal{Z}_\Gamma$, $\mathcal{Z}_\Delta$ *and* $\mathcal{Z}_\Gamma \cap \mathcal{Z}_\Delta$ satisfy conditions (a)–(c) above. Assume also that the polytopes associated with $\mathcal{Z}_\Gamma$, $\mathcal{Z}_\Delta$ and $\mathcal{Z}_\Gamma \cap \mathcal{Z}_\Delta$ are Delzant.

The idea is to use the first set of quadrics to produce a toric manifold V via symplectic reduction, and then use the second set of quadrics to define an H-minimal Lagrangian submanifold in V.

We define the real intersections of quadrics $\mathcal{R}_\Gamma$, $\mathcal{R}_\Delta$, the tori $T_\Gamma \cong \mathbb{T}^{m-n}$, $T_\Delta \cong \mathbb{T}^{m-\ell}$, and the groups $D_\Gamma \cong \mathbb{Z}_2^{m-n}$, $D_\Delta \cong \mathbb{Z}_2^{m-\ell}$ as above.

We consider the toric variety V obtained as the symplectic quotient of $\mathbb{C}^m$ by the torus corresponding to the first set of quadrics: $V = \mathcal{Z}_\Gamma / T_\Gamma$. It is a Kähler manifold of real dimension $2n$. The quotient $\mathcal{R}_\Gamma / D_\Gamma$ is the set of real points of V (the fixed point set of the complex conjugation, or the real toric manifold); it has dimension n. Consider the subset of $\mathcal{R}_\Gamma / D_\Gamma$ defined by the second set of quadrics:

$$\mathcal{S} = (\mathcal{R}_\Gamma \cap \mathcal{R}_\Delta)/D_\Gamma,$$

we have $\dim \mathcal{S} = n + \ell - m$. Finally define the n-dimensional submanifold of V:

$$N = \mathcal{S} \times_{D_\Delta} T_\Delta.$$

Theorem 3. *N is an H-minimal Lagrangian submanifold in V.*

Proof. Let $\widehat{V}$ be the symplectic quotient of V by the torus corresponding to the second set of quadrics, that is, $\widehat{V} = (V \cap \mathcal{Z}_\Delta)/T_\Delta = (\mathcal{Z}_\Gamma \cap \mathcal{Z}_\Delta)/(T_\Gamma \times T_\Delta)$. It is a toric manifold of real dimension $2(n + \ell - m)$. The submanifold of real points

$$\widehat{N} = N/T_\Delta = (\mathcal{R}_\Gamma \cap \mathcal{R}_\Delta)/(D_\Gamma \times D_\Delta) \hookrightarrow (\mathcal{Z}_\Gamma \cap \mathcal{Z}_\Delta)/(T_\Gamma \times T_\Delta) = \widehat{V}$$

is the fixed point set of the complex conjugation, hence it is a totally geodesic submanifold. In particular, $\widehat{N}$ is a minimal submanifold in $\widehat{V}$. According to [1, Cor. 2.7], N is an H-minimal submanifold in V. $\square$

Example 4.

1. If $m - \ell = 0$, i.e. $\mathcal{Z}_\Delta = \varnothing$, then $V = \mathbb{C}^m$ and we get the original construction of H-minimal Lagrangian submanifolds N in $\mathbb{C}^m$.
2. If $m - n = 0$, i.e. $\mathcal{Z}_\Gamma = \varnothing$, then N is set of real points of V. It is minimal (totally geodesic).
3. If $m - \ell = 1$, i.e. $\mathcal{Z}_\Delta \cong S^{2m-1}$, then we get H-minimal Lagrangian submanifolds in $V = \mathbb{C}P^{m-1}$. This subsumes many previously constructed families of projective examples.

References

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